On 2-fold graceful labelings of graphs

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Let \( \mathbb{Z} \) denote the set of integers and \( \mathbb{N} \) denote the set of nonnegative integers. For integers \( a \) and \( b \) with \( a \leq b \), let \([a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\} \). For a positive integer \( k \), let \( 2K_k \) denote the 2-fold complete mutigraph of order \( k \). Similarly, let \( 2[a, b] \) denote the multiset that contains every element of \([a, b]\) exactly two times. Let \( G \) be a multigraph of size \( n \), order at most \( n + 1 \), and edge multiplicity at most 2. A labeling of \( G \) is a one-to-one function \( f: V(G) \to \mathbb{N} \). If \( f \) is a labeling of \( G \) and \( e = \{u, v\} \in E(G) \), let \( \bar{f}(e) = |f(u) - f(v)| \). A 2-fold graceful labeling of \( G \) is a one-to-one function \( f: V(G) \to [0, n] \) such that:

\[
\{\bar{f}(e) : e \in E(G)\} = \begin{cases} 
2[1, \frac{n}{2}] & \text{if } n \text{ is even,} \\
2[1, \frac{n-1}{2}] \cup \{\frac{n+1}{2}\} & \text{if } n \text{ is odd.}
\end{cases}
\]

A graph \( G \) is said to be 2-fold graceful if it admits a 2-fold graceful labeling. It is known that if \( G \) with \( n \) edges is 2-fold graceful, then there exists a cyclic \( G \)-decomposition of \( 2K_{n+1} \). It is conjectured that every tree is 2-fold graceful. We investigate 2-fold graceful labelings of various classes of graphs including several classes of trees. This work was completed as part of the Illinois State University REU for Pre-service and In-service Secondary Mathematics Teachers.

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